

Scaling of entanglement entropy and superselection rules

Israel Klich^{1,2} and Leonid Levitov^{1,3}

¹ *Kavli Institute for Theoretical Physics, University of California Santa Barbara, CA 93106*

² *Department of Physics, University of Virginia, Charlottesville, VA 22904*

³ *Department of Physics, Massachusetts Institute of Technology, Cambridge MA 02139*

Particle number conservation in fermionic systems restricts the allowed local operations on bipartite systems. We show how this restriction is related to measurement entropy of particle fluctuations and compute it for several regimes of practical relevance. The accessible entanglement entropy restricted by particle number conservation is equal, to leading order, to the full entanglement entropy. The correction is bounded by the log of the variance of particle number fluctuations. The results are applied to generic Fermi systems in dimension $d > 1$ as well as to several critical systems in $d = 1$.

Entanglement entropy is a quantity which is often used in quantum information theory to characterize the degree of entanglement between different parts of a quantum system that share the same microscopic variables [1]. Prior to its adoption in the field of quantum information, entanglement entropy has been introduced and studied in field theory as a possible contribution to the entropy of black holes [2, 3, 4], where it was called “geometric entropy.” Recently it emerged as a quantity of interest in many-body theory, where it was employed as a measure of large-scale, nonlocal correlations signaling criticality [5] (for a review of these developments see Ref.[6] and references therein).

The non-local correlations between entangled quantum variables are revealed by measurements made locally on two (or more) subsystems of a quantum system. The locality of observables used to detect entanglement is of particular importance in applications such as teleportation [7] and quantum cryptography [8], which require that the two subsystems do not interact after the initial state is prepared. However, our ability to probe entanglement by a combination of local measurements is often constrained by conservation laws (e.g. of particle number, charge or spin), rendering some local observables physically inaccessible.

Constraints on the local operations (known as superselection rules) can limit experimentally accessible entanglement. The interplay of super-selection and entanglement was first analyzed in Ref.[9], where the accessible entanglement was quantified by averaging of entanglement entropy over super-selection sectors. The quantity defined in Ref.[9], which we refer to as *accessible entanglement entropy* was further discussed in Refs.[10, 11, 12]. A different point of view on accessible entanglement was put forward in Ref.[13], where super-selection constraints on local operations are treated as a resource that can be employed for hiding information in correlations which are blocked from local probing.

To understand the general relation between accessible entanglement entropy and the full entanglement entropy, in this Letter we present general results for fermion systems, interacting or noninteracting. Fermion systems

play a special role in the theory of entanglement entropy because, on one hand, they are many-body systems exhibiting rich and interesting behavior and, on the other hand, they provide a good model of experimentally relevant settings. In particular, the scaling of the full entanglement entropy in Fermi systems depends on the nature of the state. If the system is gapped, the entropy scales with the area of the boundary of the region, $S \propto L^{d-1}$, where L is the linear size of the region (*c.f.* [14, 15]). However if the system is in a gapless (metallic) state, the entropy scales as $S \propto L^{d-1} \log L$ [16, 17]. Furthermore, dynamics of fermionic systems can be used to generate complex entangled states. In particular, generation of entanglement in mesoscopic conductors occurs naturally as a result of elastic scattering of electrons on barriers and disorder potential [18, 19]. A scheme for detecting electronic entanglement which can discriminate between occupation number entanglement and mode entanglement has been proposed in [20].

One particularly attractive aspect of Fermi systems is that the full many body entanglement entropy of free fermions can be linked to experimentally accessible quantities, such as particle number fluctuations [21] or current fluctuations [22]. The latter quantity was discussed in detail in a recent proposal [22] of an experiment to measure the entanglement entropy by detecting electric noise generated in a process of connecting two Fermi seas through a quantum point contact (QPC).

While proposals such as [21, 22] as well as studies of scaling of entropy in fermion systems [16, 17] are of interest from the quantum information perspective, they do not make a distinction between entanglement entropy and the accessible entanglement entropy. Given that only the accessible entropy can be used as a source of entanglement for quantum information applications, here we set out to investigate the effect of super-selection rules on the scaling of the accessible entanglement in many-body systems of fermions. We find that, under very general assumptions, super-selection rules result in a difference between the accessible and full entanglement entropies which is small in a relative sense, provided that the number of Fermi particles contributing to the entropy is large,

$N \gg 1$. It means that, in essence, the schemes proposed for measuring the full entanglement entropy can also yield a good estimate of the accessible entanglement entropy, and vice versa.

To state our results in a quantitative form, let us recall that entanglement entropy is defined for a system partitioned into two parts A and B . The quantum state of the system is projected on A , giving the reduced density matrix $\rho_A = \text{Tr}_B \rho$, where all degrees of freedom outside A have been integrated out. Entanglement entropy is then given by the von Neumann formula: $S_A = -\text{Tr} \rho_A \log \rho_A$. When super-selection rules are present, a natural quantity to consider is S_A^{res} [9], which is obtained by averaging entanglement entropy over subspaces with fixed conserved quantity (in our case, particle number) as will be described below. We show that if the state ρ has a fixed number of particles then

$$S_A - \Delta S \leq S_A^{\text{res}} \leq S_A, \quad \Delta S = \frac{1}{2} \log [2\pi e (C_2 + \frac{1}{12})] \quad (1)$$

where $C_2 = \langle (N_A - \langle N_A \rangle)^2 \rangle$ is the variance in the number of particles in subsystem A . An immediate consequence is that since typically C_2 can not grow faster than polynomially with the size of the subregion, the correction ΔS to the entropy due to particle conservation is at most logarithmic in the volume.

In many-body systems, where entropy scaling is of interest, entanglement entropy S_A of translationally invariant systems (gapless as well as gapped) in space dimension $d > 1$ typically scales at least as the boundary area L^{d-1} , where L is the size of the region A . Given that ΔS in Eq.(1) grows at most as $\log L$, we see that the correction to the entropy from the terms violating super-selection is sub-leading to the entropy S_A . The situation may be more complicated in dimension $d = 1$, where for *critical systems*, described by a conformal field theory, entropy of a region of length L scales as $\log L$ [30]. Still, even in this case, if the fermion density-density correlations decay as $1/r^2$ or faster, the quantity $\log C_2$ grows at most as $\log \log L$, and thus ΔS is again sub-leading to S_A . Several systems of interest exhibiting this behavior are analyzed below.

Perhaps surprisingly, we found that S_A^{res} may be computed explicitly in many experimentally relevant settings. This is largely due to the fact that the difference $S_A - S_A^{\text{res}}$ is nothing but the measurement entropy S_m of particle number in subregion A . We analyze the accessible entanglement entropy for several cases: Entanglement generated when two Fermi seas are connected via a QPC for a time Δt and then disconnected, entanglement generated by a dc current in a QPC biased by voltage V during time Δt , entanglement in a Luttinger liquid and in a d -dimensional free fermion system. The results for $S_m = S_A - S_A^{\text{res}}$ are summarized in Table I.

System	$S_m = S_A - S_A^{\text{res}}$
Fermi seas connected via a QPC (Fig.1a)	$\log \log(\Delta t/\tau)$
DC voltage-biased QPC (Fig.1b)	$\log(\Delta t e V/h)$
Chiral Luttinger liquid	$\log \log(\tilde{L})$
Fermions in dimension d	$\log(\tilde{L}^{d-1} \log \tilde{L})$

TABLE I: The difference between full and accessible entropy for the systems described in the text (τ is a short time cutoff, $\tilde{L} = k_F L$, where k_F is Fermi momentum and L is system size).

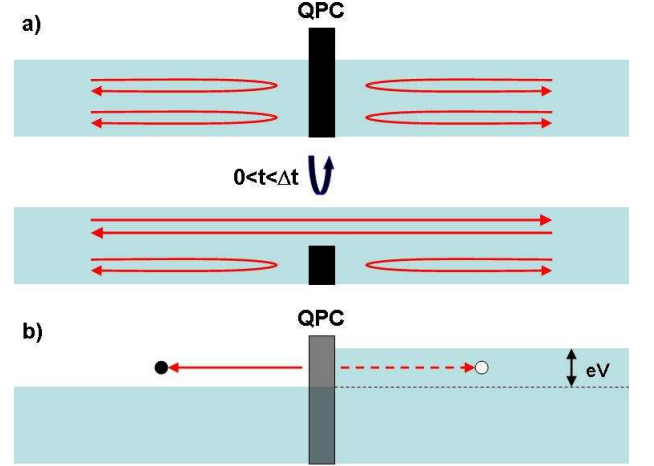


FIG. 1: Schematic many-body evolution in a quantum point contact (QPC), generating entanglement and current fluctuations. Two fermionic reservoirs are coupled through a tunable tunnel barrier, switching between fully closed and fully open states (a), or being time-independent (b). In the case (a), the left and right reservoirs, which are connected during $0 < t < \Delta t$, and disconnected at earlier and later times, are maintained at equal chemical potentials. In the case (b) the QPC is biased by a DC voltage V , with transmission fixed at a constant value $0 < D < 1$.

SUPERSELECTION AND ENTANGLEMENT FOR TWO FERMIONS

We begin with some general comments on the nature of entanglement of fermions and the effect of particle number conservation. To motivate the definition we use for the accessible entanglement entropy, we briefly discuss what entanglement of fermions means, and how conservation laws enter the picture. In that, for reader's sake, we restate some of the observations made in Ref.[9].

Fermionic wavefunctions are always antisymmetrized, but this property by itself does not imply entanglement. To illustrate the role of antisymmetrization and of particle conservation we focus on the simplest case of just two fermions. Consider a one dimensional interval $0 < x < L$, partitioned into equal halves A and B . We define $A(x) = 1$ if $0 < x < L/2$ and $A = 0$ otherwise and similarly $B(x) = 1$ if $L/2 < x < L$ and $B = 0$ otherwise.

Let us compare the properties of two antisymmetric wavefunctions,

$$\psi_1(x, y) = \frac{\sqrt{2}}{L}(A(x)B(y) - A(y)B(x)) \quad (2)$$

and

$$\psi_2(x, y) = \frac{1}{\sqrt{2}L}(e^{\frac{4\pi ix}{L}} - e^{\frac{4\pi iy}{L}}). \quad (3)$$

The state ψ_1 is obtained by putting one fermion in box A and another in box B . While this state may appear “entangled” due to the antisymmetrization, it is of course not entangled. The reduced density matrix in B is just a projection on a single particle with wavefunction $B(x)$, and thus it is a pure state. In other words, lack of information about the state in A , does not increase the entropy of the state in B . Thus, *anti-symmetrization of the wavefunction by itself does not produce entanglement*.

In contrast, the state ψ_2 , obtained by putting one particle in the state $\frac{1}{\sqrt{L}}$ and another particle in the state $\frac{1}{\sqrt{L}}e^{\frac{4\pi ix}{L}}$, contains entanglement. To see this we decompose ψ_2 as follows:

$$\psi_2(x, y) = F_{AA} + F_{BB} + F_{AB} + F_{BA} \quad (4)$$

where

$$\begin{aligned} F_{AA} &= \frac{1}{\sqrt{2}L}(e^{\frac{4\pi ix}{L}} - e^{\frac{4\pi iy}{L}})A(x)A(y) \\ F_{BB} &= \frac{1}{\sqrt{2}L}(e^{\frac{4\pi ix}{L}} - e^{\frac{4\pi iy}{L}})B(x)B(y) \\ F_{AB} &= \frac{1}{\sqrt{2}L}(e^{\frac{4\pi ix}{L}}A(x)B(y) - e^{\frac{4\pi iy}{L}}A(y)B(x)) \\ F_{BA} &= \frac{1}{\sqrt{2}L}(e^{\frac{4\pi ix}{L}}B(x)A(y) - e^{\frac{4\pi iy}{L}}B(y)A(x)) \end{aligned} \quad (5)$$

The F_{AA} and F_{BB} wavefunctions represent the possibility that both fermions are simultaneously in A or in B . Let us now measure the number of particles in A . If the result of this measurement is 0 or 2, the wavefunction is collapsed into F_{AA} or F_{BB} . However, if we find just one particle in A , we have collapsed the wavefunction into the state $F_{AB} + F_{BA}$.

The wavefunction $F_{AB} + F_{BA}$ is maximally entangled, in complete analogy with a singlet spin state of Einstein-Podolsky-Rosen form. Indeed, we can denote $|\uparrow\rangle_A$ and $|\downarrow\rangle_A$ as having a particle in A in the modes $\sqrt{\frac{2}{L}}e^{\frac{4\pi ix}{L}}A(x)$ and $\sqrt{\frac{2}{L}}A(x)$, and similarly for $|\uparrow\rangle_B$ and $|\downarrow\rangle_B$ (the form $e^{\frac{4\pi ix}{L}}$ was chosen so that ${}_A\langle\uparrow|\downarrow\rangle_A = 0$). In this notation we have

$$F_{AB} + F_{BA} = \frac{1}{\sqrt{2}}(|\uparrow\rangle_A|\downarrow\rangle_B + |\downarrow\rangle_A|\uparrow\rangle_B) \quad (6)$$

Locally measuring whether the “up” or “down” mode is occupied, and locally moving a particle between these states (for example by coupling to a potential) are valid

physical operations corresponding in spin notation to σ_z and σ_x .

This entanglement can be contrasted with the properties of the two-particle state $F_{AA} + F_{BB}$. While the state $F_{AA} + F_{BB}$ is formally entangled, this entanglement can not be revealed by Bell measurements. The states analogous to \downarrow and \uparrow here will be “having no particles in A ” and “having two particles in A ”, which could be also associated with spin variables $\sigma_z = \pm 1$ and directly measured. However, because of particle number conservation, we are blocked from rotating between these “up” and “down” states, and so σ_x is inaccessible. This blocks us from a necessary ingredient to perform Bell measurement, rendering this entanglement “unuseful”. We have thus seen that in this example, with probability 1/2, we can perform measurements that violate Bell inequalities on ψ_2 .

Consider a large number n of systems with the wavefunction ψ_2 . Let p denote the probability of measuring one charge in A in state ψ_2 . If we now measure the number of particles in A , we will average pn cases of (maximally entangled pairs) $F_{AB} + F_{BA}$. This motivates defining the entanglement of the system by averaging over the probabilities to fall in different super-selection sectors.

The discussion above, was concerned with the case of copies of a simple system. In contrast, when considering partitioning of many body systems, we are interested in *scaling* properties. Here, the entanglement content of a single system grows in a non-trivial way as the system size is increased. Thus, we are considering the entropy scaling in many-body systems, when super-selection rules are applied.

DERIVATION OF THE MAIN RESULT

We now proceed to introduce the super-selection accessible entanglement entropy. We assume that the local operations we may perform in A and B respect conservation of quantities \hat{N}_A and \hat{N}_B . The super-selection sectors are conveniently represented as the invariant subspaces of an operator $\hat{N} = \hat{N}_A \otimes I + I \otimes \hat{N}_B$ (where I is the identity operator). Thus \hat{N} represents the globally conserved quantity (i.e. total number of particles) and \hat{N}_A (\hat{N}_B) are the locally conserved quantities, which can be thought of as “number of particles in A (B)”. The available entanglement entropy is obtained by averaging over the entropy obtained by first restricting the density matrix to a sector with fixed numbers of particles N_A , N_B [9]. More concretely, this is done using the density matrices

$$\rho_{n,m} = \frac{1}{p_{n,m}} \Pi_n^A \otimes \Pi_m^B \rho \Pi_n^A \otimes \Pi_m^B, \quad (7)$$

$$p_{n,m} = \text{Tr}(\Pi_n^A \otimes \Pi_m^B \rho \Pi_n^A \otimes \Pi_m^B), \quad (8)$$

where Π_n^A are projectors onto sectors with fixed particle number n in A (i.e. all states ψ in A , such that $\hat{N}_A\psi = n\psi$) and similarly Π_m^B projects on sectors with $N_B = m$ in B .

Following Ref.[9], we define the accessible entropy as

$$S_A^{\text{res}} = - \sum p_{n,m} \text{Tr}(\rho_{n,m})_A \log(\rho_{n,m})_A \quad (9)$$

where $(\rho_{n,m})_A = \text{Tr}_B \rho_{n,m}$. In terms of the probabilities (8) we can write the Shannon entropy

$$S_m = - \sum_{n,m} p_{n,m} \log p_{n,m}, \quad (10)$$

which gives the *measurement* entropy of \hat{N}_A and \hat{N}_B [23].

We now turn to the proof of our main result (1). We assume that we know that the state of the system $|\psi\rangle$ is characterized by a fixed total number of particles, $\hat{N}|\psi\rangle = N|\psi\rangle$. The Schmidt decomposition of the state $|\psi\rangle$ may be written as $|\psi\rangle = \sum_n C_n^\alpha |n, \alpha\rangle_A \otimes |N-n, \alpha\rangle_B$, where α enumerates different states with the same quantum number, which equals n in A , and $N-n$ in B . Tracing over B we see that only the reduced density matrices $(\rho_{n,m})_A$ with $m = N-n$ are nonzero, giving

$$S_A^{\text{res}} = - \sum_n p_{n,N-n} \text{Tr}(\rho_{n,N-n})_A \log(\rho_{n,N-n})_A. \quad (11)$$

Using the constraint $n+m = N$, we can also simplify the expression (7), writing it as

$$(\rho_{n,N-n})_A = \frac{1}{p_{n,N-n}} \Pi_n^A \rho_A \Pi_n^A. \quad (12)$$

Combining this formula with the representation $\rho_A = \sum_n p_{n,N-n} (\rho_{n,N-n})_A$, and noting that $(\rho_{n,N-n})_A (\rho_{n',N-n'})_A = 0$ if $n \neq n'$, we can write the accessible entropy (9) as

$$\begin{aligned} S_A^{\text{res}} &= - \sum_n \text{Tr}(\Pi_n \rho_A \Pi_n) \log \frac{\Pi_n \rho_A \Pi_n}{p_{n,N-n}} \\ &= - \text{Tr} \rho_A \log \rho_A + \sum_n p_{n,N-n} \log p_{n,N-n} \\ &= S_A - S_m. \end{aligned} \quad (13)$$

Here $S_m = - \sum_n p_{n,N-n} \log p_{n,N-n}$ is the measurement entropy (10) for fixed total particle number $N = n+m$. The latter constraint makes particle number measurements in A and B perfectly correlated, and so particle fluctuations in A alone are sufficient to determine S_m .

The relation (13) may be interpreted as an extension of the well known relation from information theory to the quantum case: If $S(X,Y)$ is the Shannon entropy associated with the joint probability distribution of the random variables X, Y , then $S(X,Y) = S(X|Y) - S(Y)$ where $S(X|Y)$ is the entropy of X conditioned on knowing Y , and $S(Y)$ is the entropy of Y .

Next, we estimate by how much S_A^{res} can depart from S_A . This can be achieved using the relation $S_A^{\text{res}} = S_A - S_m$ derived above, and maximizing S_m . It is a basic result of information theory that given a variance C_2 and mean C_1 , a *continuous* distribution maximizing the entropy is a Gaussian distribution (this can be easily proved using Lagrange multipliers). The entropy of a Gaussian distribution, $S_0 = \frac{1}{2} \log 2\pi e C_2$, where $C_2 = \langle\langle N_A^2 \rangle\rangle$ thus supplies an upper bound on measurement entropy of a continuous observable. For a *discrete* variable, such as particle number, charge, or spin, this upper bound on entropy has to be slightly modified [24] to be

$$S_m \leq \Delta S \equiv \frac{1}{2} \log [2\pi e (C_2 + \frac{1}{12})] \quad (14)$$

establishing the inequalities (1).

It is interesting to note that in Ref. [25] the variance of particle number of noninteracting fermions was shown to provide a lower bound on the entanglement entropy, $S_A \geq (4 \log 2) C_2$. This inequality can be used to write

$$S_A - S_A^{\text{res}} \leq \frac{1}{2} \log \left[2\pi e \left(\frac{S_A}{4 \log 2} + \frac{1}{12} \right) \right] \quad (15)$$

which indicates that the difference $S_A - S_A^{\text{res}}$ becomes small in a relative sense as S_A increases.

It is straightforward to generalize the result (1) to the case of several conserved quantities, which we refer to as “charges” $a_1 \dots a_k$. For the measurement entropy $S_m(a_1 \dots a_k)$ of these quantities, using the subadditivity property of entropy, we have

$$S_m(a_1 \dots a_k) \leq S_m(a_1) + \dots + S_m(a_k).$$

Using for each of the terms the inequality (14), we arrive at

$$S_A - S_A^{\text{res}} = S_m(a_1 \dots a_k) \leq \sum_{i=1 \dots k} \frac{1}{2} \log [2\pi e (C_2(a_i) + \frac{1}{12})],$$

where $C_2(a_i)$ is the variance in measurement of the charge a_i ($i = 1 \dots k$).

MEASUREMENT ENTROPY FOR STATIC AND DYNAMICAL ENTANGLEMENT GENERATION

Now, having established the connection between accessible entanglement entropy and measurement entropy, we turn to discuss properties of the latter. In addition to being useful as a tool in providing the above bounds on S_A^{res} , the quantity S_m is directly measurable, and thus is of interest in itself.

In general, of course, the conserved charge does not have to be particle number. For example, the conserved quantity may be chosen to be energy. In this case, considered in Ref. [26], it was argued that for some situations

of interest the measurement entropy of energy S_m can mimic the behavior of thermodynamic entropy such as the second law of thermodynamics, and invariance under adiabatic evolution of the system.

A useful tool for computing S_m in many-body systems is provided by the generating function, defined as a Fourier transform of the probability distribution:

$$\chi(\lambda) = \sum_n p_{n,N-n} e^{i\lambda n} = \text{Tr} \left(e^{i\lambda \hat{N}_A} \rho \right). \quad (16)$$

Since the generating function (16) is represented as an expectation value, $\chi(\lambda) = \langle e^{i\lambda \hat{N}_A} \rangle_\rho$, it is often more easy to evaluate the quantity (16) than the probability distribution itself. For fermionic systems, in particular, this can be done with the help of the determinant representation used to analyze counting statistics of current fluctuations [27]. Once the quantity $\chi(\lambda)$ is known, the probabilities $p_{n,N-n}$, found from its Fourier transform, can be used to evaluate the measurement entropy S_m . However, if the generating function happens to be Gaussian, $\chi(\lambda) = e^{-C_2 \lambda^2/2}$, the quantity S_m can be evaluated directly as $S_m = \frac{1}{2} \log(2\pi e C_2)$ to leading order in C_2 .

Below we apply this approach to several cases of interest. We first consider the correction due to S_m for entanglement entropy generated in the process of connecting two initially separate systems [22]. The entanglement generated in this case is logarithmic in observation time. In another case, considered in [12] entanglement is generated per unit time in a nonequilibrium, but *steady state* of an open system. Here particles are transmitted across a QPC in the presence of a bias voltage. In [12] the role of super-selection by a stronger constraint, that of energy conservation, was computed, leading to a correction to entanglement entropy also scaling linearly with measurement time. We find that if only number conservation is taken into account, the correction to accessible entropy is in fact sub-leading.

We then proceed to consider the accessible entropy of critical systems. Such systems are of particular interest in the theory of entanglement entropy due to the presence of enhanced, non local, correlations [5, 16, 17, 28, 29, 30]. Here we focus on an interacting system in 1d, namely a Luttinger liquid, whose entropy can be obtained using conformal field theory methods. Somewhat surprisingly, the accessible entropy of this strongly correlated system can also be found since S_m may be evaluated using bosonization. Finally, we consider a gapless system of free fermions in higher dimensions. We show that, in an arbitrary space dimension d , the quantity S_m can be estimated using Widom's conjecture [36]. In all of the cases, S_m turns out to be a sub-leading contribution to the scaling of many-body entanglement entropy.

(1) First, we consider two Fermi seas coupled through a QPC with an externally controlled transmission coefficient (Fig. 1a). Initially the two Fermi seas are disconnected and there is no entanglement between them. The

QPC is then opened during the time interval $0 < t < \Delta t$, and then closed again. The resulting entanglement entropy was computed in [22].

The measurement entropy of particle number fluctuations the Fermi seas, which gives the correction to entanglement entropy due to charge conservation, can be evaluated as follows. We use the generating function $\chi(\lambda)$ defined by (16), where the state ρ is the density matrix of the two-lead system at the final time $t = \Delta t$, and $\hat{N}_A = \int_{x<0} \psi^\dagger(x) \psi(x) dx$ is the charge operator in the left lead. The quantity $\chi(\lambda)$ in this situation was analyzed in Ref.[22] with the help of an expression [31]

$$\chi(\lambda) = \exp \left(\frac{-\lambda_*^2 \log \frac{\Delta t}{\tau}}{2\pi^2} \right), \quad \sin \frac{\lambda_*}{2} = \sqrt{D} \sin \frac{\lambda}{2}, \quad (17)$$

where D is the QPC transmission during $0 < t < \Delta t$, and τ is a short time cutoff of the order of the time it takes to switch the QPC from a disconnected to a connected state. In this case $C_2 = \frac{D}{\pi^2} \log \frac{\Delta t}{\tau}$. For large Δt , the main contribution comes from small λ , and the distribution is Gaussian to a good approximation, in formal analogy to the Central Limit Theorem.

We therefore conclude that S_m is given by the entropy of a Gaussian probability distribution, i.e. $S_m \sim \frac{1}{2} \log 2\pi e C_2 \sim \frac{1}{2} \log \log \frac{\Delta t}{\tau}$. Since the full entanglement entropy generated in this processes is $S \sim \frac{\pi^2}{3} C_2$ [22], we see that the $S_{res} = S$ to leading order in C_2 . This is our first concrete example for our general expectation that super-selection leads to a small correction to entanglement entropy.

(2) Next, we analyze entropy generated by a voltage-biased QPC (see Fig. 1b). To leading order in the time of measurement Δt , we can characterize the systems by $N = \Delta t e V / h$ independent transmission attempts, with the probability of success equal D for each attempt. (The effects of QPC opening and closing are sub-leading since they are logarithmic in Δt , as in the previous case.) The probability to transmit n particles out of N attempts is given by

$$p_n = \frac{N!}{(N-n)!n!} D^n (1-D)^{N-n} \quad (18)$$

The large- N asymptotic form of the measurement entropy $S_m = -\sum p_n \log p_n$ of overall transmitted charge is given by [32]

$$S_m = \frac{1}{2} \log(2\pi e D(1-D)N), \quad (19)$$

whereas the entanglement entropy, generated in this process is [12]

$$S = N(D \log D + (1-D) \log(1-D)). \quad (20)$$

We see again that S_m , Eq.(19), is a sub-leading correction to the entropy (20).

One may further restrict the allowed operations by not considering entanglement between particles that cross the barrier with different energies. This restriction, which was considered in [12], substantially reduces the entropy, since the measurement entropy of energy turns out to be also linear in N .

(3) Now we shall analyze an interacting fermion systems in one dimension, described by Luttinger liquid model. Below we shall evaluate the correction to the entanglement entropy of a region A of length L due to fermion number conservation. In bosonization framework, the chiral Luttinger liquid is a conformal theory with central charge $c = 1$. Therefore its entanglement entropy scales as $\frac{1}{3} \log(L/a)$, where a is a short-distance cutoff [4]. However, since the Luttinger system is fermionic, this entanglement entropy again includes sectors which mix number of particles. Since the overall charge of the system is still a good quantum number, the relation (13) holds, and so we can estimate the difference $S_A - S_A^{\text{res}}$ by evaluating S_m for this situation. The Luttinger Hamiltonian can be written as [33]

$$H = \frac{v}{2} \int dx : \frac{1}{g} \partial_x \Theta(x)^2 + g \partial_x \phi(x)^2 :,$$

where $\Theta(x)$ and $\partial\phi(x)$ are canonical conjugates and g depends on the interaction. Within bosonization $\psi(x) \propto \sum_{n \text{ odd}} e^{-i\sqrt{\pi}\phi(x)} e^{-in(\sqrt{\pi}\Theta(x) + k_F x)}$. The density is given by $\rho = \partial_x \Theta(x) + k_F/\pi$ and so $\hat{N}_A = \Theta(l) - \Theta(0) + k_F L/\pi$.

The generating function (16) can now be computed from the bosonic theory:

$$\chi_{Lutt}(\lambda) = \langle e^{i\lambda \hat{N}_A} \rangle = e^{i\lambda k_F L/\pi - \frac{g\lambda^2}{4\pi} \log(k_F L)}, \quad (21)$$

giving a Gaussian distribution of charge. Therefore S_m for this case scales as $\log \log k_F L$.

(4) The last case we shall consider is a problem of free fermions in d dimensions. Here the state of the system is described by a Fermi sea

$$|\psi\rangle = \prod_{k \in \Gamma} a_k^\dagger |0\rangle, \quad (22)$$

where Γ is a domain in momentum space, illustrated in Fig. 2, defining the set of occupied states. The entanglement entropy of a region A in real space has been studied in [16, 17] in the asymptotic limit where the linear size of A is rescaled by a large factor L .

To compute the correction due to charge conservation we again evaluate the measurement entropy S_m using $\chi(\lambda)$. The asymptotic form of the generating function $\chi(\lambda)$ at large L can be analyzed as follows. Let P_{LA} be a projection on the region A in *real* space rescaled by a factor L (i.e. the set of points Lx where $x \in A$, as illustrated in Fig.2). Let P_Γ be a projection on the set Γ in momentum space, obtained from the Fermi distribution (22). The generating function $\chi(\lambda)$ can be written

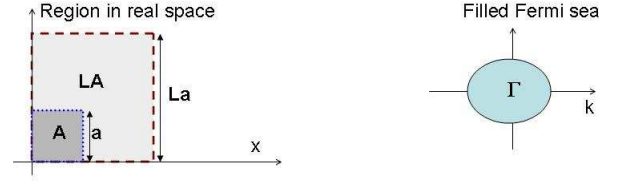


FIG. 2: The region LA , in which particle fluctuations are analyzed, is shown along with the region A of the same shape, having size of order unity. The scaling factor L is used to define scaling of particle number mean and variance, Eq.(25). A region Γ in momentum space defines the Fermi sea, Eq.(22)

as [34, 35]

$$\begin{aligned} \log \chi(\lambda) &= \log \langle e^{i\lambda \hat{N}_{LA}} \rangle = \text{Tr} \log(1 - P_\Gamma + P_\Gamma e^{i\lambda P_{LA}}) \\ &= \text{Tr} \log(1 + P_{LA} P_\Gamma P_{LA} (e^{i\lambda} - 1)) \end{aligned} \quad (23)$$

where $\hat{N}_{LA} = \int_{LA} \psi_x^\dagger \psi_x dx$.

We now estimate χ using Widom's conjecture [36]. This method has been used in [17] to estimate the scaling of entanglement of free fermions in arbitrary dimensions. While a rigorous proof for the conjecture is still missing, it seems to be perfectly consistent with numerical computations [37, 39]. Widom's conjecture [36] is a generalization of the strong Szegő theorem to higher dimensions. It states that given a function $f(z)$, which is analytic on $|z| \leq 1$, with $f(z) = 0$, the following holds as $L \rightarrow \infty$:

$$\begin{aligned} \text{Tr} f(P_\Gamma P_{LA} P_\Gamma) &= c_1 f(1) L^d + c_2 U(f) L^{d-1} \log L \\ &+ o(L^{d-1} \log L), \quad U(f) = \int_0^1 \frac{f(t) - t f(1)}{t(1-t)} dt. \end{aligned} \quad (24)$$

Here the notation $g = o(h)$ means that $g/h \rightarrow 0$ when $L \rightarrow \infty$, and the coefficients $c_{1,2}$ are given by $c_1 = \frac{1}{(2\pi)^d} \int_A \int_\Gamma dx dp$, $c_2 = \frac{\log 2}{(2\pi)^{d+1}} \int_{\partial A} \int_{\partial \Gamma} |\mathbf{n}_x \cdot \mathbf{n}_p| dS_x dS_p$, where $\mathbf{n}_x, \mathbf{n}_p$ are unit normals to $\partial A, \partial \Gamma$, respectively. The formula was proved for $d = 1$ in [38].

Plugging $f(z) = \log(1 + z(e^{i\lambda} - 1))$ in (24), we find that $U(f) = -\frac{\lambda^2}{2}$, and so

$$\log \chi(\lambda) = i\lambda c_1 L^d - \frac{\lambda^2}{2} c_2 L^{d-1} \log L. \quad (25)$$

Thus the charge distribution is Gaussian to leading order, and S_m scales as $\log(L^{d-1} \log L)$ and is again a sub-leading correction to the entropy.

In summary, particle number and charge conservation, as well as spin conservation, under some conditions, is an essential part of realistic Fermi systems. However, the existing discussions of many body entanglement of fermions [16, 17, 22] implicitly assume that such conservation laws have no direct effect on the scaling of many-body entanglement. The analysis of the measurement entropy of particle number fluctuations and of its relation to super-selection rules and entanglement entropy,

presented above, indicates that this expectation is in fact correct. We analyzed several systems of interest, including time dependent scattering problems, the ground state of one-dimensional interacting fermions (a Luttinger liquid), and a free fermion system in higher dimensions. In all those cases we found that super-selection rules yield sub-leading corrections to the entanglement entropy. We conclude that for a generic Fermi system in dimension $d > 1$ the accessible entropy is equal, to leading order in system size, to the full entanglement entropy. The same is true for critical systems in $d = 1$ described by conformal field theories with density correlator decaying as $1/r^2$.

We would like to thank F. Verstraete and C. W. J. Beenakker for useful discussions. We acknowledge support by the National Science Foundation under Grant No. PHY05-51164. The work of L. L. was supported in part by W. M. Keck Foundation Center for Extreme Quantum Information Theory.

-
- [1] C. H. Bennett, H. J. Bernstein, S. Popescu, and B. Schumacher, Phys. Rev. A **53**, 2046 (1996).
 - [2] L. Bombelli, R. K. Koul, J. Lee, and R. D. Sorkin, Phys. Rev. D **34**, 373 (1986).
 - [3] M. Srednicki, Phys. Rev. Lett. **71**, 666 (1993).
 - [4] C. Holzhey, F. Larsen and F. Wilczek, Nucl. Phys. B **424**, 443 (1994).
 - [5] G. Vidal, J. I. Latorre, E. Rico and A. Kitaev, Phys. Rev. Lett. **90**, 227902 (2003).
 - [6] L. Amico, R. Fazio, A. Osterloh, and V. Vedral, Rev. Mod. Phys. **80**, 517 (2008).
 - [7] C. H. Bennett, G. Brassard, C. Crépeau, R. Jozsa, A. Peres, and W. K. Wootters Phys. Rev. Lett. **70**, 1895 (1993).
 - [8] A. K. Ekert, Phys. Rev. Lett. **67**, 661 (1991)
 - [9] H. M. Wiseman and J. A. Vaccaro, Phys. Rev. Lett. **91**, 097902 (2003)
 - [10] M.-C. Bañuls, J. I. Cirac, and M. M. Wolf, Phys. Rev. A **76**, 022311 (2007).
 - [11] M. R. Dowling, A. C. Doherty, and H. M. Wiseman, Phys. Rev. A **73**, 052323 (2006).
 - [12] C. W. J. Beenakker, Proc. Int. School Phys. E. Fermi, Vol. 162 (IOS Press, Amsterdam, 2006), (cond-mat/0508488).
 - [13] F. Verstraete and J. I. Cirac, Phys. Rev. Lett. **91**, 010404 (2003).
 - [14] M. B. Plenio, J. Eisert, J. Dreissig, and M. Cramer, Phys. Rev. Lett. **94**, 060503 (2005).
 - [15] M. M. Wolf, F. Verstraete, M. B. Hastings and J. I. Cirac, arXiv:0704.3906
 - [16] M. M. Wolf, Phys. Rev. Lett. **96**, 010404 (2006).
 - [17] D. Gioev and I. Klich, Phys. Rev. Lett. **96**, 100503 (2006).
 - [18] C. W. J. Beenakker, C. Emary, M. Kindermann, J. L. van Velsen, Phys. Rev. Lett. **91**, 147901 (2003).
 - [19] P. Samuelsson, E. V. Sukhorukov, and M. Büttiker, Phys. Rev. Lett. **91**, 157002 (2003)
 - [20] V. Giovannetti, D. Frustaglia, F. Taddei, and R. Fazio, Phys. Rev. B **75**, 241305 (2007).
 - [21] I. Klich, G. Refael and A. Silva, Phys. Rev. A **74**, 032306 (2006).
 - [22] I. Klich and L. S. Levitov, preprint arXiv:0804.1377
 - [23] R. Balian, Eur. J. Phys **10**, 208 (1989).
 - [24] Theorem 9.7.1 in: T. M. Cover and J. A. Thomas, *Elements of Information Theory* (Wiley, New York, 1991).
 - [25] I. Klich, J. Phys. A **39**, L85 (2006).
 - [26] R. Barankov and A. Polkovnikov, arXiv:0806.2862
 - [27] L. S. Levitov and G. B. Lesovik, JETP Lett. **58**, 230 (1993).
 - [28] G. Refael and J. E. Moore, Phys. Rev. Lett. **93**, 260602 (2004).
 - [29] B. Q. Jin and V. Korepin, J. Stat. Phys. **116**, 79 (2004)
 - [30] P. Calabrese and J. Cardy, J. Stat. Mech. P06002 (2004).
 - [31] L. S. Levitov, G. B. Lesovik, arXiv:cond-mat/9401004
 - [32] P. Jacquet and W. Szpankowski, IEEE Trans. Inform. Theory **45**, 1072 (1999).
 - [33] C. L. Kane, M. P. A. Fisher, Phys. Rev. Lett. **68**, 1220 (1992).
 - [34] I. Klich, in: “Quantum Noise in Mesoscopic Physics”, eds. Yu. V. Nazarov and Ya. M. Blanter (Kluwer 2003), cond-mat/0209642.
 - [35] M. Budde and K. Mølmer, Phys. Rev. A **70**, 053618 (2004).
 - [36] H. Widom, Operator Theory: Adv. Appl., **4**, p. 447 (Birkhäuser, Basel-Boston, 1982).
 - [37] T. Barthel, M.-C. Chung, and U. Schollwöck, Phys. Rev. A **74**, 022329 (2006).
 - [38] H. J. Landau and H. Widom, J. Math. Anal. Appl. **77**, 469 (1980).
 - [39] W. Li, L. Ding, R. Yu, T. Roscilde and S. Haas, Phys. Rev. B **74**, 073103 (2006).